**Integral Equations**

**Integral equation:** An equation containing an unknown function under one or more integral signs is called an integral equation. The general form of an integral equation is

 (1)

where  is an unknown function, and  are both known functions, and  are limits of integration, is a constant parameter.

Also  is known as the kernel of the integral equation (1).

There are two types integral equation such as:

1. Linear integral equation
2. Non-linear integral equation.

**Linear integral equation:** An integral equation is called linear if only linear operations are performed in it upon the unknown function. The most general type of linear integral equation is of the form



where  is the unknown function,  , and  are known functions.

Example: 1. 

2. .

**Non-linear integral equation:** An integral equation is called non-linear if the unknown function appears under an integral sign to a power  . The most general type of non-linear integral equation is of the form



where  is the unknown function,  , and  are known functions.

Example: 1. 

2. .

Again the linear integral equations are classified into two basic types:

1. Volterra integral equation
2. Fredholm integral equation.

**Volterra integral equation:** An integral equation is called Volterra integral equation if the upper limit of the integration is a variable. The most general type of Volterra integral equation is of the form



where  is the unknown function,  , and  are known functions.

**Case-01:** If , then



This is called the Volterra’s integral equation of first kind.

**Case-02:** If , then



This is called the Volterra’s integral equation of second kind.

**Case-03:** If and , then



This is called the homogeneous Volterra’s integral equation of second kind.

**Fredholm integral equation:** An integral equation is called Fredholm integral equation if the domain of the integration is fixed i.e. upper limit and lower limit are constant. The most general type of Fredholm integral equation is of the form



where  is the unknown function,  , and  are known functions.

**Case-01:** If , then



This is called the Fredholm integral equation of first kind.

**Case-02:** If , then



This is called the Fredholm integral equation of second kind.

**Case-03:** If and , then



This is called the homogeneous Fredholm integral equation of second kind.

**Singular integral equation:** An integral equation is called Singular integral equation if one or both limits of integration become infinite or the kernel becomes infinite at one or more points within the range of integration.

Example: 1. 

2. .

**Special kinds of kernels:** Some kernel are as follows:

1. **Symmetric kernel:** A kernel  is called symmetric (or complex symmetric or Hermitian) if



where the bar denotes the complex conjugate.A real kernel  is symmetric if .

1. **Difference kernel:** If the kernel  is dependent solely on the difference ,

i.e. 

then  is called the difference kernel.

1. **Separable or Degenerate kernel:** If the kernel  can be expressed as the sum of a

finite number of terms, each of which is the product of a function of  only and a function of  only, i.e.



then the kernel is called a separable or degenerate kernel.

1. **Iterated kernels:** (a) Consider the Volterra integral equation of second kind

.

Then, the iterated kernels  are defined as



and 

(b) Consider the Fredholm integral equation of second kind

.

Then, the iterated kernels  are defined as



and 

1. **Resolvent kernel or Reciprocal kernel:** Suppose the solution of the Volterra integral equation of second kind



is of the form 

and the solution of the Fredholm integral equation of second kind



is of the form .

Then  is called the resolvent kernel or reciprocal kernel of the given equations.

**Eigen values and Eigen functions:** Let us consider the homogeneous Fredholm integral equation

. (1)

The values of  for which (1) has a non-zero solution are called the eigen values of (1) or of the kernel  and for every eigen value of , the corresponding non-zero solution of (1) is called the eigen function.

**Question-01:** Explain the differentiation of a function under an integral sign.

**Solution:** Consider the function defined by the relation

where is a positive integral and is a constant.

Here .

Differentiating (1) w. .r. to  under the integral sign, we get

From (1), for , we have

Differentiating (2) successively times, we have

In particular, taking in (4) we get

Using (2) in (5) we get

Thus, we have

In general, we have

From (1) and (7), we have

This may be represented as the result of integrating the function from to and then integrating times, we have

(**Complete**)

**Note: Leibnitz’s Rule:** If  then the derivative w.r.to  is

.

**Question-02:** Establish the relation between differential and integral equations.

**Solution:** Thereis a fundamental relationship between integral equation and ordinary and partial differential equation with given initial values. Consider the equation

with continuous coefficients . The initial conditions are prescribed as follows:

where a prime denotes differentiation with regard to .

Consider .

Taking into account the initial conditions (2), we get

where represents for a multiple integral of order .

From (9) and (11), we have

where

Putting in (12) we get

which represents the Volterra’s integral equation of second kind.

**Question-03:** Show thatthe IVP is equivalent to the integral equation +.

**Solution:** Given that

with

From (1) we get,

Integrating both sides within to we have

## 

This is an integral equation and the given equation is equivalent to this equation. (**Showed**)

**Question-04:** State and prove the set of conditions that ensure the existence of a unique solution of the Volterra Integral Equation of the second kind:

.

**Solution: Statement:** (i) if , is real and continuous on ,

(ii) is real and continuous function on

(iii) satisfies in the Lipschitz condition then there exist continuous solution of

defined for, with

**Proof:** By the method of successive approximations, we construct the sequence as follows:

Since all are continuous functions on with given by (2).

To prove the uniform convergence of the sequence , we shall consider the associated series

Putting , in (4) we get

To obtain the estimates for

We get from (4)

Putting in (7) we get

Putting in (7) we get

Thus, according to mathematical induction we get

which holds for all positive integer .

Taking summation of (10) over we get

This series is convergent for all values , and As a result, the series (5) is absolutely and uniformly convergent on and so the sequence , since

Let

The function is continuous on and satisfies (1). In (4) we can make and obtain (1), because

and this relation holds, since

.

This shows that there exists a solution of (1).

**Uniqueness:** to prove the uniqueness, let be another continuous solution of (1) on the same interval.

Then

Now subtracting (4) from (11) we get

From (11) we have

Using this result in (12) we get

This result tends to zero as

Thus the solution is unique.

Hence the proof is complete.

**Question-05:** State and prove a set of conditions for which Fredholm linear integral equation , has a unique solution

**Solution: Statement:** Considerthe Fredholm integral equation of second kind as

where,

1. the kernel , is real and continuous in the rectangle , for which and
2. , is real and continuous in an interval , for which and is a constant.

Then the Fredholm integral equation in (1) has a unique solution in . The solution is the series

which is absolutely and uniformly convergent.

**Proof:** From (1) we have

Replacing by we get

Again replacing by in (3) we get

Putting the value of in (1) we get,

Rewriting (4) by using , and then replacing by we get

and

Putting the value of from (7) in (5) we get,

Proceeding in this way up to n times we get

where

Let us now consider the infinite series

By conditions (i) and (ii) each term of the series (11) is continuous in . Thus, the series (11) is continuous in , provided it converges uniformly in .

Let and contains the maximum value in and respectively.

Assume

Then

It will converge only if

Therefore the series (11) converges absolutely and uniformly.

If (1) has a continuous solution, it must be expressed by (9). If is continuous in , then must have a maximum value, say Thus, from (10),

Thus, satisfying (9) is the continuous function given by the series (11). This prove our desired results.

**Question-06:** Solve Volterra integral equation of second kind by the method of successive substitutions.

**Answer:** Consider the Volterra’s integral equation of second kind as

 (1)

where (i) The kernel  is real and continuous in the rectangle .

Consider , where is the maximum value in .

(ii) The function  is real and continuous in an interval .

Consider , where is the maximum value in the interval .

(iii)  is non-zero numerical parameter.

Substituting the unknown function under an integral sign from the relation (1), we get





Performing this operation successively for , we have







In general, we have







(2)

Now consider the infinite series

 (3)

Let 

Then 

Since  and , then





It follows that the series is convergent for all values of and hence the series (3) is absolutely and uniformly convergent.

Again, 

.

Therefore, we notice that the function , which satisfies the relation (2), is the continuous function given by the infinite series (3), the integral equation (1) has a unique continuous solution in the interval .

**Question-07:** Solve the non-homogeneous Volterra integral equation of second kind by the method of successive approximations.

**Answer:** Consider the Volterra’s integral equation of second kind as

 (1)

where the kernel  is continuous function for  and  is continuous for .

Consider an infinite power series in  as,

 (2)

Let the series (2) is a solution of the integral equation (1), then





Equating the coefficients of like powers of , we get

 (3)

 (4)

 (5)



 (6)

This yields a method for a successive approximation of the function , where the series (2) converges uniformly in  and  for any .

Now from (3) and (4), we get



From (5), we get



By interchanging the order of integration, we get

; 



where 

In general,  ;  (7)

The functions are called iterated kernels.

So that 

Therefore, , are defined by

;  (8)

Now from the relation (2), we get









where 

The function is called resolvent kernel or reciprocal lernel of the integral equation (1).

Thus, the solution of the integral equation (1) is given by

.

**Question-08:** Solve Fredholm integral equation of second kind by the method of successive substitutions.

**Answer:** Consider the Fredholm integral equation of second kind as

 (1)

where (i) The kernel  is real and continuous in the rectangle .

Consider , where is the maximum value in .

(ii) The function  is real and continuous in an interval .

Consider , where is the maximum value in the interval .

(iii)  is non-zero numerical parameter.

Substituting the unknown function under an integral sign from the relation (1), we get





Performing this operation successively for , we have







In general, we have







(2)

Now consider the infinite series

 (3)

Let 

Then 

Since  and , then



It will converge only if

 (4)

Thus, the series (2) converges absolutely and uniformly when the relation (3) holds.

Again, 

.

Therefore, the function , which satisfies the relation (2), is the continuous function given by the infinite series (3), the integral equation (1) has a unique continuous solution in the interval .

**Question-09:** Solve the Fredholm integral equation of second kind by the method of successive approximations.

**Answer:** Consider the Fredholm integral equation of second kind as

 (1)

where (i) The kernel  is real and continuous in the rectangle .

(ii) The function  is real and continuous in an interval .

(iii)  is non-zero numerical parameter.

Consider an infinite power series in  as,

 (2)

Let the series (2) is a solution of the integral equation (1), then



 (3)

Equating the coefficients of like powers of , we get

 (4)

 (5)

 (6)



 (7)

This yields a method for a successive approximation of the function .

Now from (4) and (5), we get



From (6), we get









where 

In general,  ;  (7)

The functions are called iterated kernels.

So that 

Therefore, , are defined by

;  (8)

Now from the relation (2), we get









where 

The function is called resolvent kernel or reciprocal lernel of the integral equation (1).

Thus, the solution of the integral equation (1) is given by

.

**Question-10:** Show that all iterated kernels of symmetric kernels are also symmetric and the eigenvalues of a symmetric kernel are real.

**Answer:** The iterated kernel is defined as

Since the kernel is symmetric so and so on.

Now substituting , , , , for , , , , we get

Thus, the kernel is also symmetric.

Hence by induction, we conclude that every iterated kernel of a symmetric kernel is symmetric.

**2nd part:** We know that if and are distinct eigen values of distinct eigen functions and then

Let and

Then and

Using these values on (1) we get

Hence the eigen values of a symmetric kernel are real. (**Showed**).

**Question-11:** Prove that an initial value problem is equivalent to an integral equation.

**Solution:** Let us consider the following initial value problem

with

From (1) we get

Integrating (3) within the limit to we get

which is an integral equation.

Thus, the initial value problem is equivalent to an integral equation. (**Proved**)

**Problem**

**P-01:** Show that , is a solution of the VIE .

**Solution:** Given that,

Substituting the function in the right hand side of (1), we have

Hence is a solution of the given Volterra’s integral equation. (**Showed**)

P**-02:** Show that is a solution of.

**Solution:** Given that,

Substituting the function in the right hand side of (1), we have

Let

Now

Using (3) in (2) we get

Hence is a solution of the given equation. (**Showed**)

**P-03:** Show that the function is a solution of the integral equation

**Solution:** Given that

Now

Thus, the function satisfies the given integral equation. Hence is the solution of that equation. (**Showed**)

P**-04:** Show that  is a solution of the integral equation .

**Solution:** Given that,  (1)

Substituting the function  in the left hand side of (1), we have

















Hence  is a solution of the given equation. (**Showed**).

P**-05:** Verify that whether  is a solution of the equation .

**Solution:** Given that,  (1)

Substituting the function  in equation (1), we have



















Hence  is not a solution of the given integral equation. (**Verified**).

P**-06:** Verify that whether  is a solution of the integral equation 

**Solution:** Given that,  (1)

Substituting the function  in equation (1), we have















Hence  is a solution of the given integral equation. (**Verified**).

P**-07:** Show that the function  is a solution of the equation .

**Solution:** Given that, 

 (1)

Substituting the function  in equation (1), we have















Hence  is a solution of the given integral equation. (**Showed**).

P**-08:** Show that the function  is a solution of the Fredholm integral equation



where 

**Solution:** Given that,  (1)

where 

Replacing  by , we get



Substituting the function  in equation (1), we have

















Hence  is a solution of the given integral equation. (**Showed**).

P**-09:** Show that the function  is a solution of the Fredholm integral equation



where 

**Solution:** Given that,  (1)

where 

Replacing  by , we get



Substituting the function  in equation (1), we have























Hence  is a solution of the given integral equation. (**Showed**).

**P-10:** Transform the IVP , ,  into an integral equation.

**Solution:** Given initial value problem is



 (1)

,  (2)

Consider

 (3)

Integrating (3) within the limit  to , we get







 (4)

Again integrating (4) within the limit  to, we get







 (5)

Using the values of (3), (4) and (5) in (1) we get





This is the required integral equation, which represents the Volterra’s integral equation of second kind.

**P-11:** Transform the IVP , ,  into an integral equation.

**Solution:** Given initial value problem is

 (1)

,  (2)

Consider

 (3)

Integrating (3) within the limit  to , we get







 (4)

Again integrating (4) within the limit  to, we get







 (5)

Using the values of (3), (4) and (5) in (1) we get





This is the required integral equation, which represents the Volterra’s integral equation of second kind.

**P-12:** Transform the IVP , ,  into an integral equation.

**Solution:** Given initial value problem is

 (1)

,  (2)

Consider

 (3)

Integrating (3) within the limit  to , we get







 (4)

Again integrating (4) within the limit  to, we get







 (5)

 (6)

Again, differentiating (5) within the limit to , we get







 (7)

Using the values of (3), (4) and (7) in (1) we get





This is the required integral equation, which represents the Volterra’s integral equation of second kind.

**P-13:** Transform the IVP into an integral equation and solved it.

**Solution:** Given initial value problem is

Consider

Integrating (3) w.r.to ‘x’ within the limit to , we get

Again integrating (4) w.r.to ‘x’ within the limit to , we get

Using the values of (3), (4) and (5) in (1) we get

where , and .

This is the required transform integral equation, which represents the Volterra’s integral equation of second kind.

**2nd part:** Now we have to find out the solution of (6) in the form of an infinite series of .

Let

From (6) and (7) we have

Equating the coefficients of same power of in both side, we get

Now

Using these values in (7) we get

This is the required solution of the integral equation obtained from the given ordinary differential equation.

**P-14:** Convert the IVP into an integral equation and hence solve it.

**Solution:** Given initial value problem is

and

Consider

Integrating (3) w.r.to ‘x’ within the limit to , we get

Again integrating (4) w.r.to ‘x’ within the limit to , we get

Using the values of (3), (4) and (5) in (1) we get

where , and .

This is the required integral equation, which represents the Volterra’s integral equation of second kind.

**2nd part:** Let

From (6) and (7) we get

Equating the coefficients of same power of in both side, we get

Now

Using these values in (7) we get

This is the required solution.

**P-15:** Convert the initial value problem to an integral equation.

**Solution:** Given initial value problem is

with

From (1), we get

Integrating (3) within the limit to we get

This is the required integral equation.

**P-16:** Convert the initial value problem into a Volterra Integral Equation of second kind.

**Solution:** Given initial value problem is

and

Consider

Integrating (3) w.r.to ‘x’ within the limit to , we get

Again integrating (4) w.r.to ‘x’ within the limit to , we get

Using the values of (3), (4) and (5) in (1) we get

where , and .

This is the required transform integral equation, which represents the Volterra’s integral equation of second kind.

**P-17:** Verify that is a solution of the IVP .

**Solution:** Given IVP is,

and

Also the given function is

Differentiating (3) with respect to *x* we get

Putting these values in (1) we get

Hence is a solution of the given initial value problem. (**Verified**)

**P-18:** Convert the IVP into an integral equation.

**Solution:** Given IVP is

and

Integrating (1) w.r.to ‘x’ within the limit to x, we get

Again integrating (3) w.r.to ‘x’ within the limit to x, we get

This is the required integral equation, which represents the Volterra’s integral equation of second kind.

**P-19:** Solve the VIE: .

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

and the solution of (2) is

Comparing (2) with (1) we have

, , and .

The Resolvent kernel is

and

Putting in (5) we get

Putting in (5) we get

Putting in (5) we get

Now, we assume that

Now from (4), we have

Now from (3), the solution is

This is the required solution. (**Solved**)

**P-20:** Solve: .

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

The solution of (2) is

Comparing (2) with (1), we have

, , and

Also we have the Resolvent kernel

and

Putting in (5) we get

Putting in (5) we get

Putting in (5) we get

Now, we assume that

Now from (4) we have

Now from (3), the solution is

This is the required solution. (**Solved**)

**P-21:** Solve: and verify your result.

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

and the solution of (2) is

Comparing (2) with (1) we have

, , and .

The Resolvent kernel is

and

Putting in (5) we get

Putting in (5) we get

Putting in (5) we get

Now, we assume that

Now from (4) we have

Now from (3), the solution is

This is the required solution. (**Solved**)

**Verification:** We have

Putting in the right side of (1) we get

(**Verified**)

**P-22:** Find the resolvent kernel of the VIE .

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

and the solution of (2) is

Comparing (2) with (1) we have

, , and .

The resolvent kernel is

and

Putting in (5) we get

Putting in (5) we get

Putting in (5) we get

Now, we assume that

Now from (4) we have

This is the required resolvent kernel. (**Ans**)

**P-23:** Find the resolvent kernel and hence solve the integral equation .

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

and the solution of (2) is

Comparing (2) with (1) we have

, , and .

The resolvent kernel is

and

Putting in (5) we get

Putting in (5) we get

Putting in (5) we get

Now, we assume that

Now from (4) we have

This is the required resolvent kernel.

From (3) we get

This is the required solution.

**P-24:** Solve the VIE: .

**Solution:** Given that

We know that the Volterra’s integral equation of second kind is

and the solution of (2) is

Comparing (2) with (1) we have

, , and .

The Resolvent kernel is

and (5)

Putting in (5) we get













Again,





Similarly,





The resolvent kernel is







The solution of the given integral equation is













This is the required solution. (**Solved**)

**P-25:** Solve the VIE: 

**Solution:** Given that  (1)

We know that the Volterra’s integral equation of second kind is

 (2)

and the solution of (2) is

 (3)

Comparing (2) with (1) we have

, , and .

The Resolvent kernel is

and (5)

Putting in (5) we get





Again,











Similarly,





The resolvent kernel is







The solution of the given integral equation is













This is the required solution. (**Solved**)

**P-26:** Solve .

**Solution:** Given that

where

Using (1) in (2) we get

Putting the value of in (1) we get

(**Ans**)

**P-27:** Solve: and verify your result.

**Solution:** Given that

where

From (1) and (2), we get

Putting the value of in (1) we get

where (**Ans**)

**Verification**:

L.H.S

Now

R.H.S

Since L.H.S = R.H.S. Hence the result is verified.

**P-28:** Solve the IE .

**Solution:** Given that

where

Substituting (2) into (3), we get

and

Now

Putting these values in (4) and (5) we get

and

Thus, we get a non-homogeneous system of linear equations

The determinant of this system is

Thus, the system has a unique solution.

The solution is given by

Using these values in (1) we get

(**Ans**)

**P-29:** Find the eigen values and eigen functions of

**Solution:** Given that

where

and

Substituting (2) into (3) we get,

and

Now

Put

when then

when then

Using these in the above integral we get

Putting these values in (4) and (5) we get

and

Thus, we get a homogeneous system of linear equations

The determinant of the eigen values is

Thus, the eigen values are .

When then (8) reduces to

is arbitrary and

From (2), the eigen function is

When then (8) reduces to

and is arbitrary

From (2), the eigen function is

Thus, the eigen values are and the corresponding eigen functions are

(**Ans**)

**P-30:** Find the eigen values and eigen functions of

**Solution:** Given that

where

Substituting (2) into (3) we get,

and

Now

Putting these values in (4) and (5) we get

and

Thus, we get a homogeneous system of linear equations

The determinant of the eigen values is

Thus, the eigen values are .

When then (8) reduces to

and is arbitrary.

From (2), the eigen function is

When then (8) reduces to

is arbitrary and .

From (2), the eigen function is

Thus, the eigen values are and the corresponding eigen functions are

(**Ans**)

**P-31:** Find the eigen values and eigen functions for

**Solution:** Given that

where

Substituting (2) into (3) we get,

and

Now

Putting these values in (4) and (5) we get

and

Thus, we get a homogeneous system of linear equations

The determinant of the eigen values is

Thus, the eigen values are .

When then (8) reduces to

and is arbitrary.

From (2), the eigen function is

When then (8) reduces to

is arbitrary and .

From (2), the eigen function is

Thus, the eigen values are and the corresponding eigen functions are

(**Ans**)

**P-32:** Find the eigen values and eigen functions of

**Solution:** Given that

where

Substituting (2) into (3) we get,

And

Finally,

Thus, we get a homogeneous system of linear equations

The determinant of the eigen values is

Thus, the eigen value is .

When then Eq. (4) reduces to

Eq. (5) reduces to

And Eq. (6) reduces to

Taking we get and

Putting the values of , , and in Eq. (2) we get

Thus, the eigen value is and the corresponding eigen function is

(**Ans**)

**P-33:** Find the Resolvent kernel and hence solve the Fredholm integral equation 

**Solution:** Given that  (1)

We know that the Fredholm integral equation of second kind is

 (2)

and the solution of (2) is

 (3)

Comparing (2) with (1) we have

, , ,and .

The Resolvent kernel is

 (4)

where  (5)

and 

Putting  in (5) we get











Again,











Similarly,





The resolvent kernel is









The solution of the given integral equation is









This is the required solution. (**Solved**)

**P-34:** Solve the Fredholm integral equation 

**Solution:** Given that  (1)

We know that the Fredholm integral equation of second kind is

 (2)

and the solution of (2) is

 (3)

Comparing (2) with (1) we have

, , , and .

The Resolvent kernel is

 (4)

where  (5)

and 

Putting  in (5) we get











Again,











Similarly,







The resolvent kernel is













The solution of the given integral equation is

















This is the required solution. (**Solved**)

**P-35:** Find the Resolvent kernel of 

**Solution:** Given that  (1)

The Resolvent kernel is

 (2)

where  (3)

and 

Putting  in (3) we get











Again,











Similarly,







The resolvent kernel is









.

**P-36:** Find the Iterated kernel of 

**Solution:** Given that 

By the definition of Iterated kernel, we get























Similarly, 





The required Iterated kernel is

.

**P-37:** Find the Iterated kernel of 

**Solution:** Given that 

By the definition of Iterated kernel, we get



















Similarly, 



if , then

 for 

if , then

 for 